<u>Plan</u> §1. Introduction
§2. Equivariant (co)homology group
§3. Instanton partition function
§4. Heisenberg / Virasoro algebra action on the equiv. cohomology group of instanton moduli spaces

§1

AIM : Study of cohomology groups of moduli spaces of sheaves on $\mathbb{A}^2$

- integral of cohomology classes
  - $\longrightarrow$ instanton partition functions (originated in physics)
  - $\longrightarrow$ application to enumerative geometry e.g. Donaldson = Seiberg–Witten

- convolution operators
  - $\longrightarrow$ representation theory of ∞-dim'l algebras

§2. Equivariant (co)homology group

<u>2.1</u> X: variety with $\mathbb{G}_m$ (more generally T) action

$\Longrightarrow$ $H^*_{\mathbb{G}_m}(X)$, $H_*^{\mathbb{G}_m}(X)$ ($\mathbb{C}$-coefficients)

satisfying functorial properties as usual (co)homology groups

Take $V = \mathbb{C}^N \curvearrowleft \mathbb{G}_m$ $(N \gg 0)$

Note • $V \setminus \{0\} \curvearrowleft \mathbb{G}_m$ free

• $H^i(V\setminus\{0\}) = H^i(S^{2N-1}) = 0$ except $i = 0, 2N-1$

Consider $X_V := X \times (V\setminus\{0\})/\mathbb{G}_m \longleftarrow X \times (V \setminus \{0\})$, and

principal $\mathbb{G}_m$-b'dle

set $H^i_{\mathbb{G}_m}(X) := H^i(X_V)$.

○ independence of V if $\dim V \gg 0$

$\because$ $X_{V_1} \longleftarrow X \times (V_1\setminus\{0\}) \times (V_2 \setminus \{0\}) / \mathbb{G}_m \longrightarrow X_{V_2}$

fiber bundles with fibers $V_2\setminus\{0\}$, $V_1\setminus\{0\}$ respectively.

no cohomology except 0 & very large

$\Rightarrow H^i(X_{V_1}) \cong H^i(\text{middle}) \cong H^i(X_{V_2})$ //

○ $f: X \to Y$ $\mathbb{G}_m$-equivariant morphism

$\Longrightarrow f^*: H^*_{\mathbb{G}_m}(Y) \longrightarrow H^*_{\mathbb{G}_m}(X)$

○ Suppose $X \curvearrowleft \mathbb{G}_m$ free. Then $X_V \longrightarrow X/\mathbb{G}_m$ is a fiber b'dle with fiber $V\setminus\{0\}$ no cohomology except $0, 2N-1$.

$\Longrightarrow H^i_{\mathbb{G}_m}(X) \cong H^i(X/\mathbb{G}_m)$

○ E: $\mathbb{G}_m$-equivariant vector bundle $\Longrightarrow c_i(E) \in H^{2i}_{\mathbb{G}_m}(X)$

equivariant Chern class

○ $H^*_{\mathbb{G}_m}(X)$ has the cup product $H^i_{\mathbb{G}_m}(X) \otimes H^j_{\mathbb{G}_m}(X) \longrightarrow H^{i+j}_{\mathbb{G}_m}(X)$
$f^*: H^*_{\mathbb{G}_m}(Y) \longrightarrow H^*_{\mathbb{G}_m}(X)$ is a ring hom.

<u>Ex.</u> $X = pt$ L: canonical 1 dim rep. of $\mathbb{G}_m$
$\longrightarrow$ equivariant vector b'dle over X
$X_V = \mathbb{P}(V)$: $L_V = \mathcal{O}(1)$ over X_V
Let $\hbar := c_1(L_V) \in H^2(X_V)$
Then $H^*(X_V) \cong \mathbb{C}[\hbar]/(\hbar^N = 0) \overset{N\to\infty}{\rightsquigarrow} H^*_{\mathbb{G}_m}(pt) \cong \mathbb{C}[\hbar]$.

○ $X \longrightarrow pt$ $\Rightarrow$ $H^*_{\mathbb{G}_m}(pt) \longrightarrow H^*_{\mathbb{G}_m}(X)$

$\therefore$ $H^*_{\mathbb{G}_m}(X)$ is a module over $H^*_{\mathbb{G}_m}(pt) = \mathbb{C}[\hbar]$

○ $X_V = X \times (V \setminus \{0\})/\mathbb{G}_m$ $\Longrightarrow$ $H^*_{\mathbb{G}_m}(X) \longrightarrow H^*(X)$
$\downarrow$ fiber X
$\mathbb{P}(V)$ restriction to the fiber (forgetting hom)

<u>Rem</u> torus case : $T = \underbrace{\mathbb{C}^* \times \cdots \times \mathbb{C}^*}_{\ell}$ $\Rightarrow$ Use $V = \mathbb{C}^{N\ell}$: T-module

$H^*_T(pt) = \mathbb{C}[\hbar_1, \cdots, \hbar_\ell] = \mathbb{C}[\mathrm{Lie}\, T]$ (polynomial on Lie T)

$T' \subset T$ subtorus $\Longrightarrow$ $H^*_T(X) \longrightarrow H^*_{T'}(X)$ restriction hom

Given by pull-back

$X \times (V \setminus \{0\})/_T \longleftarrow X \times (V' \setminus \{0\})/_{T'}$

In particular, we have
$$\begin{array}{ccc} H^*_T(pt) & \longrightarrow & H^*_{T'}(pt) \\ \| & & \| \\ \mathbb{C}[\mathrm{Lie}\, T] & \longrightarrow & \mathbb{C}[\mathrm{Lie}\, T'] \end{array}$$

This is induced by $\mathrm{Lie}\, T' \longrightarrow \mathrm{Lie}\, T$.

2.2 Next we introduce equivariant homology groups.

We will use the fundamental class $[X]$ for a possibly noncompact variety X.

$\longrightarrow$ need to use Borel-Moore homology groups (homology groups of locally finite chains)

X: smooth $\Rightarrow$ Poincaré duality $H^i(X) \xrightarrow{\cong} H^{BM}_{2\dim X - i}(X)$ ← denoted by $H_*(X)$ hereafter

$\alpha \longmapsto \alpha \cap [X]$

NB $[X]$ is defined even when X is not smooth.

Define $H_i^{\mathbb{G}_m}(X) := H_{i+2(N-1)}(X_{\mathbb{V}})$ $\qquad 2(N-1) = \dim(\mathbb{V}\setminus\{0\})/\mathbb{G}_m$
$= \dim X_{\mathbb{V}} - \dim X$

- cap product $H^i_{\mathbb{G}_m}(X) \otimes H_j^{\mathbb{G}_m}(X) \longrightarrow H^{\mathbb{G}_m}_{j-i}(X)$; $\alpha \otimes c \mapsto \alpha \cap c$

$\therefore$ $H_*^{\mathbb{G}_m}(X)$ is a module over $H^*_{\mathbb{G}_m}(X)$ (and hence also over $H^*_{\mathbb{G}_m}(pt)$)

- X: smooth $\Longrightarrow$ $H^i_{\mathbb{G}_m}(X) \cong H^{\mathbb{G}_m}_{2\dim X - i}(X)$

$\alpha \mapsto \alpha \cap [X]$

- $f: X \to Y$ proper $\mathbb{G}_m$-equivariant morphism

$\Longrightarrow f_*: H_*^{\mathbb{G}_m}(X) \longrightarrow H_*^{\mathbb{G}_m}(Y)$

- $X \curvearrowleft \mathbb{G}_m$ free $\Longrightarrow$ $H_i^{\mathbb{G}_m}(X) \cong H_{i-2}(X/\mathbb{G}_m)$

$[X] \mapsto [X/\mathbb{G}_m]$

2.3 localization thm

Consider $H^*_T(X)$, $H^T_*(X)$ as modules over $H^*_T(pt) = \mathbb{C}[\mathrm{Lie}\,T]$

Suppose $\mathrm{Stab}\,x = T'$ $\forall x \in X$ $\quad X \curvearrowleft T/T'$ free

<u>Claim</u> $\mathrm{Supp}\, H^*_T(X) \subset \mathrm{Lie}\,T'$ $(\subset \mathrm{Lie}\,T)$

(∵) $X_V = X \times V\setminus\{0\}/T = X \times \left(V\setminus\{0\}/T'\right)/T/T' \xrightarrow[\text{fiber} \,=\, V\setminus\{0\}/T']{} X/_{T/T'}$

∴ locally $X_V = V\setminus\{0\}/T' \times X/_{T/T'}$

∴ $X_V \longrightarrow V\setminus\{0\}/T$ factors through $V\setminus\{0\}/T'$

$$\begin{tikzcd}
H^*(V\setminus\{0\}/T) \arrow[rr] \arrow[d, "V\to\infty"'] \arrow[dr] & & H^*(X_V) \\
\mathbb{C}[\mathrm{Lie}\,T] & H^*(V\setminus\{0\}/T') \arrow[ur] \arrow[d] & \\
 & \mathbb{C}[\mathrm{Lie}\,T'] &
\end{tikzcd}$$

//

Generally $X \curvearrowleft T$ can be stratified by Stab

$\mathrm{Stab}\,x = T \iff x$: fixed pt

∴ $X \setminus X^T$ has stab $\subsetneq T$

$\Longrightarrow \mathrm{Supp}\, H^*_T(X\setminus X^T) \subsetneq \mathrm{Lie}\,\mathbb{C}$

<u>Th.</u> (localization) $\quad H^*_T(X) \xrightarrow{i^*} H^*_T(X^T) \cong H^*_T(pt) \otimes H^*(X^T)$

is an isomorphism over the generic point of $\mathrm{Lie}\,T$,

i.e. $\otimes_{\mathbb{C}[\mathrm{Lie}T]} \mathbb{C}(\mathrm{Lie}\,T)$ ← quotient field

The same is true for $H^T_*(X^T) \xrightarrow{i_*} H^T_*(X)$

2.4 Fixed point formula

Assume X : nonsingular

$\therefore$ Poincaré duality $H_T^*(X) \cong H^T_{2\dim X - *}(X)$

X^T : also nonsingular

$= \coprod X_\alpha$: connected component

$$i^* i_* : \underset{\substack{\text{\rotatebox{90}{$\cong$}} \\ H_T^*(pt)\otimes H_*(X^T)}}{H_*^T(X^T)} \longrightarrow H_*^T(X^T) \quad \text{preserves } H_*^T(X_\alpha).$$

Let $i_\alpha = i|_{X_\alpha}$

<u>Prop</u> $i_\alpha^* i_{\alpha *}\Big|_{H_*^T(X_\alpha)} = e(N_\alpha) \frown \cdot$

N_α = normal bdle, $e(N_\alpha)$ = equivariant Euler (top Chern) class

(proof) In the neighbourhood of X_α, X

$$\begin{array}{ccc} X & \cong & N_\alpha \\ \cup & & \cup \ \text{0-section} \\ X_\alpha & = & X_\alpha \end{array}$$

$\therefore$ enough to compute $i^* i_*$ for $i_\alpha : X_\alpha \longrightarrow N_\alpha$.

Then $i_\alpha^* i_{\alpha *} = e(N_\alpha) \frown \cdot$ //

<u>Lemma</u>. $e(N_\alpha)$ is invertible in $H_*^T(X_\alpha) \otimes_{H_*^T(pt)} \mathbb{C}(\mathrm{Lie}\, T)$

$\because$ $H_T^*(X_\alpha) = H_T^*(pt) \otimes H^*(X_\alpha)$

$H_T^*(pt) \otimes H^{>0}(X_\alpha)$ nilpotent as $H^{>2\dim X_\alpha}(X_\alpha) = 0$.

$\therefore$ enough to check that the $H_T^*(pt)\otimes H^0(X_\alpha) \cong \mathbb{C}[\mathrm{Lie}\, T]$ component of $e(N_\alpha) \neq 0$

Take $x \in X_\alpha$ $\quad T_x X \curvearrowleft T$-module $= \bigoplus V_\lambda$

$\lambda : T \to \mathbb{C}^*$ weight $\quad V_\lambda = \{ v \in T_x X \mid {}^\forall t \cdot v = \lambda(t) v \}$

Then $T_x X_\alpha = V_1$, $N_\alpha = \bigoplus_{\lambda \neq 1} V_\lambda$

$\mathbb{C}[\mathrm{Lie}\,T]$-component of $e(N_\alpha) = \prod_{\lambda \neq 1} d\lambda^{\dim V_\lambda}$

where $d\lambda : \mathrm{Lie}\,T \longrightarrow \mathbb{C} \quad \in \mathbb{C}[\mathrm{Lie}\,T]$

Since $\lambda \neq 1 \Rightarrow d\lambda \neq 0$, therefore $\prod \neq 0$ //

<u>Th</u> Assume X : nonsingular and proper $a : X \longrightarrow pt$

$\omega \in H_*^T(X)$

$$\mathbb{C}[\mathrm{Lie}\,T] \ni \int_X \omega = a_* \omega = \sum_\alpha \int_{X_\alpha} e(N_\alpha)^{-1} i_\alpha^* \omega \in \mathbb{C}(\mathrm{Lie}\,T)$$

$\because$ $$a_* \omega = a_* \sum_\alpha i_{\alpha *} i_{\alpha *}^{-1} \omega = \sum_\alpha \underbrace{(a \circ i_\alpha)_*}_{= \int_{X_\alpha} \cdot} i_\alpha^{-1} \omega$$

$$i_\alpha^* \omega = i_\alpha^* i_{\alpha *} i_{\alpha *}^{-1} \omega = e(N_\alpha) \cap i_{\alpha *}^{-1} \omega$$

$\therefore \quad i_{\alpha *}^{-1} \omega = e(N_\alpha)^{-1} i_\alpha^* \omega$ //